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CONDUCTIVE HEAT TRANSFER TO AN ARBITRARILY SHAPED BODY IN A VARIABLE PROPERTY FLUID

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NOMENCLATURE

A, B, C, coefficients defined in equation (1);
f, nondimensional heat conductivity σ/σ_∞;
F, G, quantities dependent on temperature-dependence of σ and on body shape, respectively, defined in equation (9);
H, antiderivative of f, defined in equation (5);
L, major-to-minor axis ratio of prolate spheroid;
n, outward distance normal to body surface;
Nu, Nusselt number;
Pr, Prandtl number;
q, heat flux;
r, nondimensional radius R/R_b;
R, radius;
Re, Reynolds number based on diameter;
s, nondimensional radial coordinate defined following equation (7);
S, body surface area;
T, temperature.

Greek symbols

$\alpha, \beta, \delta, \gamma$, exponents defined in equation (1);
σ, thermal conductivity;
μ, viscosity;
τ, nondimensional temperature T/T_∞;
ξ, nondimensional radial coordinate defined following equation (7);
ω, power-law exponent.

Subscripts

b, at the body surface;
m, at the mean temperature $\frac{1}{2}(T_b+T_\infty)$;
∞, at infinite distance from the body surface;
0, for a constant-property medium;
ω, for a "power-law" medium with $\sigma \propto T^{\omega}$.

1. INTRODUCTION

THE THEORY of steady-state heat transfer to a solid body in a variable-property (temperature-dependent) fluid is still in a relatively primitive stage. For situations involving forced convection, variable-property theories exist only for small Reynolds numbers Re (Hanna and Myers [1], Chang [2], Kassoy *et al.* [3], Kassoy [4]). For Re of order unity or larger, theories exist only for constant-property fluids. These include: Garner and Keey [5] (sphere: $Re \leqslant 900$); Dennis *et al.* [6] (cylinder: $Re \leqslant 20$); Gupalo and Ryazantsev [7] (sphere: $Re \leqslant 40$); Dennis *et al.* [8] (sphere: $Re \leqslant 20$), Sucker and Brauer [9] (cylinder: $Re \leqslant 120$). An unpublished numerical treatment by Chang and Laframboise ([10], chapter 7) (sphere: $Re \leqslant 60$; cylinder: $Re \leqslant 100$) is also available. Numerous measurements of heat transfer to bodies in fluids have been made; these have been reviewed by Whitaker [11]. Such measurements are always influenced to some extent by variable-property effects, and a partial compensation for such effects is often made by defining local and total Nusselt numbers Nu_b and Nu, as well as Re, in terms of thermal conductivity σ and viscosity μ at the intermediate temperature $T_m = \frac{1}{2}(T_b+T_\infty)$ (Collis and Williams [12]), where T_b and T_∞ are body-surface and free-stream temperatures, respectively. For clarity in presenting our results (Section 2), we shall instead define these in terms of fluid properties at T_∞. Experimental and numerical results in forced-convection situations are often fitted by empirical relations such as (Whitaker [11]):

$$Nu = A + (BRe^{\alpha} + CRe^{\beta})Pr^{\delta}(\mu_\infty/\mu_b)^{\gamma} \tag{1}$$

where: $Nu = (1/S_b) \oint Nu_b \, dS_b$ evaluated over the body surface, $Nu_b = 2R_b q_b/[\sigma_m(T_\infty - T_b)]$, R_b is body radius (for a body not having circular symmetry, $2R_b$ is replaced by a characteristic body dimension), q_b is local heat flux to the body surface, and $S_b = \oint dS_b$. For convenience we shall replace S_b by $4\pi R_b^2$ in the definition of Nu, regardless of body shape. If a relation similar to (1) is used to represent Nu_b, then the corresponding coefficients A_b, B_b, C_b, α_b, β_b, δ_b, γ_b will be functions of surface position. In the nonflowing limit $ReSc \to 0$, we have $Nu \to A$ in (1). However, the value of A will in general be influenced by the form of the $\sigma(T)$ dependence. This situation is not always recognized; see e.g. Whitaker [11], equations (30)–(33). For a sphere, the above definitions, together with the usual solution of Laplace's equation, imply $A \to 2$ in the constant-property limit. For a prolate spheroid of equatorial radius R_b and major-to-minor axis ratio L, one obtains (Section 2), in the same limit:

$$A = 2(L^2-1)^{1/2}\Big/\left\{\ln\left[\frac{(L+1)^{1/2}+(L-1)^{1/2}}{(L+1)^{1/2}-(L-1)^{1/2}}\right]\right\}$$
$$\to 2L/\ln(2L) \text{ as } L \to \infty. \tag{2}$$

The latter result is often applied as an approximation for long finite cylinders. In Section 2, we shall study the departure of A from these values, by developing a calculation for Nu which is valid for arbitrary temperature dependence of σ and for arbitrary body shape.

2. THEORY

In the nonflowing (conductive) limit $RePr \to 0$, and in time-independent conditions, the nondimensional energy equation and boundary conditions become:

$$\tilde{\nabla}\cdot[f(\tau)\tilde{\nabla}\tau] = 0 \tag{3}$$

$$\left.\begin{array}{ll}\tau = 1 & \text{at infinite distance}\\ \tau = \tau_b & \text{at the body surface}\end{array}\right\} \tag{4}$$

where $\tau = T/T_\infty$, $f(\tau) = \sigma(\tau)/\sigma_\infty$, and $\tilde{\nabla} = R_b\nabla$. We introduce a new variable

$$H(\tau) = \int_1^{\tau} f(\tau')\,d\tau'. \tag{5}$$

Then $f(\tau) = \mathrm{d}H/\mathrm{d}\tau$, and (3) and (4) become:

$$\tilde{\nabla}^2 H = 0 \tag{6}$$

$$\left.\begin{aligned} H &= 0 \text{ at infinite distance} \\ H &= H(\tau_b) \equiv H_b \text{ at the body surface.} \end{aligned}\right\} \tag{7}$$

The change of variable introduced in (5) is known as the Kirchhoff transformation [4; 13, pp. 10–11; 14, pp. 353–356]. With this change of variable, all of the standard techniques for solving Laplace's equation can now be applied. It is noteworthy that the boundary condition (7b) could be trivially generalized to nonuniform surface temperatures if desired. Furthermore, (5) remains valid for all physically realistic forms of $f(\tau)$ including those having discontinuities, so (6) and (7) are applicable to conductive heat transfer even in the presence of phase changes. This would for example permit solutions, including exact analytic solutions in certain geometries, for heat transfer from objects immersed in multiphase media, and heated nonuniformly. Later we shall confine ourselves to investigation of a more restricted class of problems.

In any given problem, solution of (6) with (7) will define a set of isothermal surfaces (Laplace equipotentials) on each of which H as well as τ is constant. Such a solution will have the general form $H = H(\xi)$ where ξ is a nondimensional "radial" coordinate which is constant over each isothermal surface. The particular choice of ξ is a matter of convenience. We also choose another nondimensional coordinate $s = s(\xi)$ having the property that $H(\xi)$ is a linear function of s. We also choose $s = 0$ at infinite distance. If n_b is outward distance normal to the body surface, the local Nusselt number is now given by:

$$\begin{aligned} Nu_b &= \frac{2R_b}{T_\infty - T_b} \frac{\sigma_b}{\sigma_\infty} \frac{\partial T}{\partial n_b} \\ &= \frac{2R_b f(\tau_b)}{1-\tau_b} \frac{\mathrm{d}\tau}{\mathrm{d}H} \frac{\mathrm{d}H}{\mathrm{d}s} \frac{\mathrm{d}s}{\mathrm{d}\xi} \frac{\partial \xi}{\partial n_b}\bigg|_{\xi=\xi_b} \\ &= \frac{2R_b}{1-\tau_b} \frac{H_b - H_\infty}{s_b} \frac{\mathrm{d}s}{\mathrm{d}\xi} \frac{\partial \xi}{\partial n_b}\bigg|_{\xi=\xi_b} \\ &= \frac{2}{1-\tau_b}\left[\int_{\tau_b}^{1} f(\tau)\,\mathrm{d}\tau\right]\left(-\frac{1}{s}\frac{\mathrm{d}s}{\mathrm{d}\xi}\frac{\partial \xi}{\partial \tilde{n}_b}\right)\bigg|_{\xi=\xi_b} \end{aligned} \tag{8}$$

where $\tilde{n}_b = n_b/R_b$.

If the body surface is isothermal, we have $\partial\xi/\partial n_b \equiv \mathrm{d}\xi/\mathrm{d}n_b$, and the total Nusselt number then is:

$$Nu = \underbrace{\frac{2}{1-\tau_b}\left[\int_{\tau_b}^{1} f(\tau)\,\mathrm{d}\tau\right]}_{F}\underbrace{\left(-\frac{1}{s}\frac{\mathrm{d}s}{\mathrm{d}\xi}\right)\bigg|_{\xi=\xi_b} \frac{1}{4\pi R_b^2} \oiint \frac{\mathrm{d}\xi}{\mathrm{d}\tilde{n}_b}\,\mathrm{d}S_b}_{G}. \tag{9}$$

Equation (9) is of product form, with the dependence of Nu on conductivity variation in F, separated from its dependence on body shape in G. The same separation also applies to (8) when the body surface is isothermal. This leads to an important conclusion: the influence of conductivity variation on conductive heat transfer must have the same form for all body shapes. Equivalently: the ratio of conductive heat transfer to any two different shapes will retain the same value in the presence of arbitrary conductivity variation as in the constant-property case. In the constant-property limit, $F \to 2$. For spheres, we choose $\xi \equiv r = R/R_b$ and $s = 1/r$; then $s_b = 1$ by definition of r. We then obtain $G = 1$. For prolate spheroids, we define coordinates in terms of the usual cylindrical coordinates (ρ, ψ, z) using the relations: $\rho = a \sinh\xi \sin\eta$, $z = a\cosh\xi\cos\eta$; $\xi \geqslant 0$, $0 \leqslant \eta \leqslant \pi$; here, ρ and z represent nondimensional distances. The scale factors for these coordinates are: $h_\xi = h_\eta = a(\cosh^2\xi - \cos^2\eta)^{1/2}$; $h_\psi = a \sinh \xi \sin \eta$. Solving Laplace's equation yields $s = \ln\coth\frac{1}{2}\xi$; $\xi_b = \coth^{-1} L$. We choose $a = \operatorname{csch}\xi_b = (L^2-1)^{1/2}$; then R_b is the equatorial body radius. Evaluation of G in (9) then leads to a result which is equal to one-half the RHS of (2).

Table 1. Power-law exponents ω, and ranges of availability of experimental results fitted by power-law relations, for temperature dependence of heat conductivity of various fluids

Name of liquid	ω	Temperature range (K)
Freon CCl_2F_2	−1.023	290–330
Methyl chloride CH_3Cl	−2.20	300–330
SO_2	−0.844	250–330
Engine oil (unused)	−0.253	273–433
Name of gas		
Ne	0.676	230–330
H_2	0.69	300–1400
He	0.73	200–5000
N_2	0.73	250–2000
Air	0.78	250–2000
CO	0.83	220–450
Ar	0.88	230–330
O_2	0.89	200–600
Fluorine	0.906	200–370
NO	0.94	144–370
D_2	0.95	270–330
Methane	1.096	140–370
HCl	1.10	230–310
Cl	1.25	230–330
NH_3	1.28	223–473
Steam	1.36	250–900
CO_2	1.4	220–600
N_2O	1.49	230–330
Acetylene	1.56	200–370
Ethylene	1.71	200–370
n-Butane	1.77	260–370
i-Butane	1.77	260–370
Ethane	1.82	200–370

We now consider the special case of power-law media: $\sigma(T) \propto T^\omega$. Many materials have thermal conductivity behavior which can be approximated by power laws over useful temperature ranges. We have fitted power-law relations to some available thermal-conductivity data (Touloukian *et al.*

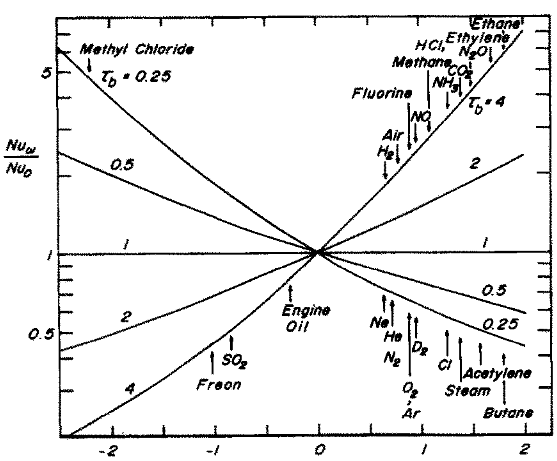

FIG. 1. Local or total heat-transfer function $F/2 = Nu_{\omega b}/Nu_{0b} = Nu_\omega/Nu_0$ as given by equation (10), for an isothermal object of arbitrary shape in the conductive (zero convection) limit, as a function of power-law exponent ω in the temperature dependence of heat conductivity σ, for various nondimensional body surface temperatures $\tau_b = T_b/T_\infty$. Here Nu_{0b} and Nu_0 are the local and total constant-property Nusselt numbers.

[15]; Eckert and Drake [16], Tables B3, B4); Table 1 summarizes the results. In nondimensional form, this relation becomes $f = \tau^{\omega}$; substitution of the latter into (9) yields:

$$F = \frac{2(1-\tau_b^{\omega+1})}{(\omega+1)(1-\tau_b)}; \quad \omega \neq -1$$
$$F = \frac{2}{1-\tau_b}\ln(\tau_b^{-1}); \quad \omega = -1. \tag{10}$$

Equation (10) therefore gives the dependence of both total and local heat transfer on ω, for an object of any shape having an isothermal surface. Figure 1 shows this dependence for various values of τ_b.

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A CONSTANT HEAT FLUX MODEL OF THE EVAPORATING INTERLINE REGION

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NOMENCLATURE

$\bar{A}$,	dispersion constant [J];
h,	heat-transfer coefficient [W · m^{-2} · K^{-1}];
h_{fg},	latent heat of vaporization [J · kg^{-1}];
k,	thermal conductivity [W · m^{-1} · K^{-1}];
M,	molecular weight [kg · mol^{-1}];
$\dot{m}$,	mass flux [kg · m^{-2} · s^{-1}];
P,	pressure [N · m^{-2}];
Q,	heat transferred [W · m^{-1}];
q,	heat flux [W · m^{-2}];
R,	universal gas constant [J · mol^{-1} · K^{-1}];
S,	solid thickness [m^{-1}];
T,	temperature [K];
U,	overall heat-transfer coefficient, [W · m^{-2} · K^{-1}];
V,	molar volume [m^3 · mol^{-1}].

Greek symbols

γ,	dimensionless heat-transfer coefficient;
δ,	film thickness [m];
η,	dimensionless film thickness;
ν,	kinematic viscosity [m^2 · s^{-1}];
ξ,	dimensionless film length coordinate;
σ,	evaporation coefficient [dimensionless].

Subscripts and superscripts

l,	liquid phase;
lv,	liquid–vapor interface;
s,	solid;
t,	constant temperature solution (1);
0,	evaluated at interline;
v,	vapor phase;
–,	averaged;
id,	ideal;
*,	differentiation with respect to ξ.

1. INTRODUCTION

In a previous paper [1], a procedure to determine the theoretical heat-transfer coefficient for the interline region (junction of vapor, evaporating thin film and non-evaporating thin film) of a wetting film was developed. Since the analysis was based on the assumption of a constant liquid–vapor interfacial temperature, the effect of the thermal conductivity of the solid on the process was not included. In this communication an approximate procedure to determine the heat sink capability of the interline region is developed in which the effect of the solid resistance to heat transfer is included. The formulation uses a simple one-dimensional constant heat flux model. Although the general approach